RELEVANCE OF NUCLEON SPIN IN AMPLITUDE ANALYSIS OF REACTIONS $\pi^- p \to \pi^0 \pi^0 n$ AND $\pi^- p \to \eta\eta n$.

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Abstract

The measurements of reactions $\pi^- p \to \pi^- \pi^+ n$ and $\pi^+ n \to \pi^+ \pi^- p$ on polarized targets at CERN found a strong dependence of pion production amplitudes on nucleon spin. Analyses of recent measurements of $\pi^- p \to \pi^0 \pi^0 n$ reaction on unpolarized targets by GAMS Collaboration at 38 GeV/c and BNL E852 Collaboration at 18 GeV/c use the assumption that pion production amplitudes do not depend on nucleon spin, in conflict with the CERN results on polarized targets. We show that measurements of $\pi^- p \to \pi^0 \pi^0 n$ and $\pi^- p \to \eta\eta n$ on unpolarized targets can be analysed in a model independent way in terms of 4 partial-wave intensities and 3 independent interference phases in the mass region where S- and D-wave dominate. We also describe model-independent amplitude analysis of $\pi^- p \to \pi^0 \pi^0 n$ reaction measured on polarized target, both in the absence and in the presence of G-wave amplitudes. We suggest that high statistics measurements of reactions $\pi^- p \to \pi^0 \pi^0 n$ and $\pi^- p \to \eta\eta n$ be made on polarized targets at Protvino

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IHEP and at BNL, and that model-independent amplitude analyses of this polarized data be performed to advance hadron spectroscopy on the level of spin dependent production amplitudes.

I. INTRODUCTION

The dependence of hadronic reactions on nucleon spin was discovered by Owen Chamberlain and his group at Berkeley in 1957 in measurements of polarization in pp and np elastic scattering at 320 MeV [1]. The prevalent belief in 1950's and 1960's was that in hadronic reactions spin is irrelevant and the spin effects observed by Chamberlain were expected to vanish at very high energies, such as 6 GeV/c. Instead, measurements of polarization in two body reactions found significant dependence on spin up to 300 GeV/c at CERN [2] and Fermilab [3]. Measurements at BNL found large spin effects at very large momentum transfers [4,5]. Inclusively produced hyperons show large polarizations up to the equivalent of 2000 GeV/c [6]. Large spin effects in inclusive reactions were observed at Fermilab Spin Facility with polarized proton and antiproton beams at 200 GeV/c [7,8]. Today, work is in progress to study dependence of hadronic reactions on spin and nucleon spin structure with polarized colliding proton beams at RHIC collider at BNL [9].

The most remarkable feature of hadronic reactions is the conversion of kinetic energy of colliding hadrons into the matter of produced particles. This conversion process is characterized by conservation of total four momentum and quantum numbers such as electric charge, baryon number and strangeness. The conversion process depends also on the flavour content and spin of colliding hadrons.

The simplest production processes are single-pion production reactions such as $\pi N \to \pi^+\pi^- N$ and $KN \to K\pi N$. In 1978, Lutz and Rybicki showed [10] that measurements of these reactions on polarized target yield enough observables that model independent amplitude analysis is possible determining the spin dependent production amplitudes. The measurements of these reactions on polarized targets are thus of special interest because they permit to study the spin dependence of pion creation directly on the level of production amplitudes. Several such measurements were actually done at CERN-PS.

The high statistics measurement of $\pi^- p \to \pi^-\pi^+ n$ at 17.2 GeV/c on unpolarized target [11] was later repeated with a transversely polarized proton target at the same energy [12–17].

Model independent amplitude analyses were performed for various intervals of dimeson mass at small momentum transfers $-t = 0.005 - 0.2$ (GeV/c)^2 [12–15], and over a large interval of momentum transfer $-t = 0.2 - 1.0$ (GeV/c)^2 [16,17].

Additional information was provided by the first measurement of $\pi^+ n \to \pi^+\pi^- p$ and $K^+ n \to K^+\pi^- p$ reactions on polarized deuteron target at 5.98 and 11.85 GeV/c [18,19]. The data allowed to study the t-evolution of mass dependence of moduli of amplitudes [20]. Detailed amplitude analyses [21,22] determined the mass dependence of amplitudes at larger momentum transfers $-t = 0.2 - 0.4$ (GeV/c)^2.

The crucial finding of all these measurements was the strong dependence of production amplitudes on nucleon spin. The process of pion production is very closely related to nucleon transversity, or nucleon spin component in direction perpendicular to the production plane. For instance, in $\pi^- p \to \pi^-\pi^+ n$ at small t and dipion masses below 1000 MeV, all amplitudes with recoil nucleon transversity down are smaller than the transversity up amplitudes, irrespective of dimeson spin and helicity. All recoil nucleon transversity down amplitudes also show suppression of resonance production in the ρ meson region.

The measurements of $\pi N \to \pi^+\pi^- N$ reactions on polarized target also enabled a model-independent separation of S- and P-wave amplitudes. The S-wave amplitude with recoil nucleon transversity up is found to resonate at 750 MeV in both solutions [23–25] irrespective of the method of amplitude analysis [25]. The resonance is narrow and the most recent fits [25] determined its width to be 108 ± 53 MeV.

Recently high statistics measurements of $\pi^- p \to \pi^0\pi^0 n$ reaction were made at 38 GeV/c by the GAMS Collaboration at IHEP Protvino [26–28] and at 18 GeV/c by the E852 Collaboration at BNL [29]. In principle one expects these experiments to confirm the existence of $\sigma(750)$ state and to search for new states in higher partial waves. However the situation is not so simple. The reason is that both groups analyse their well acquired data using a strong simplifying assumption that the production amplitudes are independent of nucleon spin [30–34]. The purpose of this assumption is to reduce the number of unknown amplitudes by one half and to enable to proceed with amplitude analysis using such spin independent

"amplitudes".

At this point it is important to realize that one does not really make an assumption that production amplitudes are independent on nucleon spin. It is a well-known fact that nucleon helicity nonflip and flip amplitudes have entirely different t-dependence due to conservation of angular momentum. The helicity flip amplitudes vanish as $t \to 0$ while helicity nonflip amplitudes do not. The model independent amplitude analyses of two-body reactions also found that the zero structure of flip and nonflip amplitudes are dramatically different. Moreover, the pion production at small t proceeds mostly via the pion exchange which contributes to helicity flip amplitudes. Thus the assumption that is really being made is that all nonflip amplitudes vanish.

The assumption that production amplitudes in $\pi^- p \to \pi^0 \pi^0 n$ do not depend on nucleon spin is in conflict with the general consensus that hadronic reactions depend on nucleon spin up to the highest energies, and contradicts all that we have learned from measurements of $\pi N \to \pi^- \pi^+ N$ on polarized targets at CERN. Applied to reactions $\pi^- p \to \pi^+ \pi^- n$ and $\pi^+ n \to \pi^+ \pi^- p$, the assumption has observable consequences that can be tested directly in measurements with polarized targets.

The first consequence is that all polarized moments p^L_M vanish identically. All experiments on polarized targets however found large nonzero polarized moments. An example is given in Fig. 1 which shows polarized target asymmetry A related to the moment p^0_0. The polarized target asymmetry has large nonzero (negative) values in both reactions. Measurements of $K^+ n \to K^+ \pi^- p$ show similarly large values of A [19].

The experiments on polarized targets are best analysed using nucleon transversity amplitudes rather than nucleon helicity amplitudes. The second consequence of the assumption of independence of production amplitudes on nucleon spin is that all transversity amplitudes $|\overline{A}|$ with recoil nucleon transversity "up" are equal in magnitude to transversity amplitudes $|A|$ with recoil nucleon transversity "down" relative to the scattering plane $\pi^- N \to (\pi^- \pi^+) N$. In Fig. 2 we show the ratios of transversity amplitudes for S-, P-, D- and F-waves for dimeson helicity $\lambda = 0$. The ratios are far from unity, indicating that production amplitudes

depend strongly on nucleon spin.

If the assumption that the production amplitudes are independent of nucleon spin does not work in reactions $\pi^- p \to \pi^- \pi^+ n$, $\pi^+ n \to \pi^+ \pi^- p$ and $K^+ n \to K^+ \pi^- p$ then there is no reason to assume that it will work in $\pi^- p \to \pi^0 \pi^0 n$ reaction. We must conclude that some of the results of the analyses of $\pi^- p \to \pi^0 \pi^0 n$ by GAMS and E852 collaborations are not reliable.

The question of reliability of amplitude analyses based on assumption of independence of production amplitudes on nucleon spin is of special importance to confirmation and further study of the narrow $\sigma(750)$ state in $\pi^- p \to \pi^0 \pi^0 n$ reaction. The evidence for narrow $\sigma(750)$ is closely connected to the spin dependence of production amplitudes. In Fig. 3 we show the two S-wave production amplitudes for $\pi^- p \to \pi^- \pi^+ n$. We see that while the transversity up amplitude $|\overline{S}|^2 \Sigma$ resonates in both solutions around 750 MeV the transversity down amplitude $|S|^2 \Sigma$ is large and non-resonating. This results in a partial wave intensity $I_S = (|S|^2 + |\overline{S}|^2)\Sigma$ that does not necessarily show a narrow resonant behaviour. As seen in Fig. 4, such is the case of solution $I_S(2,2)$.

It is therefore necessary to establish what quantities can be determined from the measurements of $\pi^- p \to \pi^0 \pi^0 n$ on unpolarized targets without the assumption of independence of production amplitudes on nucleon spin. Furthermore, it is necessary to find out if a model independent amplitude analysis of $\pi^- p \to \pi^0 \pi^0 n$ in measurements on polarized targets is possible. The purpose of this work is to provide answers to these questions. We shall show that in measurements of $\pi^- p \to \pi^0 \pi^0 n$ on unpolarized targets in the region where S- and D-wave dominate, one can measure four spin-averaged partial wave intensities and three unrelated phases connected with the spin-averaged interference terms. We will also show that model independent amplitude analysis is possible when measurements of $\pi^- p \to \pi^0 \pi^0$ are made on polarized target, both in the region where S- and D-wave dominate as well as in the region where G-wave also contributes. We shall propose that such measurements are a natural extension of measurements on unpolarized targets and should be performed at both IHEP in Protvino and at BNL using Brookhaven Multi Particle Spectrometer.

The paper is organized in seven sections. The kinematics, observables and pion production amplitude are introduced in Section II. The method of model independent analysis of data on unpolarized target is described in Section III. In Section IV we compare this method with model dependent analyses of GAMS and E852 Collaborations. In Section V we describe a model-independent amplitude analysis of $\pi^- p \to \pi^0\pi^0 n$ on polarized target in the absence of G-wave. In Section VI we extend the model-independent amplitude analysis to include the G-wave amplitudes. The paper closes with a summary and proposals for measurements of $\pi^- p \to \pi^0\pi^0 n$ and $\pi^- p \to \eta\eta n$ on polarized targets in Section VII.

II. KINEMATICS, OBSERVABLES, AND AMPLITUDES

A. Kinematics

Various aspects of phase space, kinematics and amplitudes in pion production in $\pi N \to \pi\pi N$ reactions are described in several books [35–37]. The kinematical variables used to describe the dimeson production on a polarized target at rest are $(s, t, m, \theta, \phi, \psi, \delta)$ where s is the c.m.s. energy squared, t is the four-momentum transfer to the nucleon squared, and m is the dimeson invariant mass. The angles (θ, ϕ) describe the direction of π^0 in the $\pi^0\pi^0$ rest frame. The angle ψ is the angle between the direction of target transverse polarization and the normal to the scattering plane (Fig. 5). The angle δ is the angle between the direction of target polarization vector and its transverse component (projection of polarization vector into the x, y plane). The analysis is usually carried out in the t-channel helicity frame for the $\pi^0\pi^0$ dimeson system. The helicities of the initial and final nucleons are always defined in the s-channel helicity frame.

B. Observables

In our discussion of observables measured in $\pi^- p \to \pi^0\pi^0 n$ with polarized targets we follow the notation of Lutz and Rybicki [10]. When the polarization of the recoil nucleon is

not measured, the unnormalized angular distribution $I(\theta,\phi,\psi,\delta)$ of $\pi^0\pi^0$ (or $\eta\eta$) production on polarized nucleons at rest of fixed s, m and t can be written as

$$I(\Omega,\psi,\delta) = I_U(\Omega) + P_T \cos\psi I_C(\Omega) + P_T \sin\psi I_S(\Omega) + P_L I_L(\Omega) \tag{2.1}$$

where $P_T = P\cos\delta$ and $P_L = P\sin\delta$ are the transverse and longitudinal components of target polarization $\vec{P}$ with respect to the incident momentum (Fig. 5). The simple $\cos\psi$ and $\sin\psi$ dependence is due to spin $\frac{1}{2}$ of the target nucleon [10,38]. Parity conservation requires I_U and I_C to be symmetric, and I_S and I_L to be antisymmetric in ϕ. In the data analysis of angular distribution of the dimeson system, it is convenient to use expansions of the angular distributions into spherical harmonics. In the notation of Lutz and Rybicki we have

$$I_U(\Omega) = \sum_{L,M} t^L_M \mathrm{Re}Y^L_M(\Omega) \tag{2.2}$$

$$I_C(\Omega) = \sum_{L,M} p^L_M \mathrm{Re}Y^L_M(\Omega)$$

$$I_S(\Omega) = \sum_{L,M} r^L_M \mathrm{Im}Y^L_M(\Omega)$$

$$I_L(\Omega) = \sum_{L,M} q^L_M \mathrm{Im}Y^L_M(\Omega)]$$

The expansion coefficients t, p, r, q are called multipole moments. The moments t^L_M are unpolarized. The moments p^L_M, r^L_M and q^L_M are polarized moments. Experiments with transversely polarized targets measure only transverse moments p^L_M and r^L_M but not the longitudinal moments q^L_M.

The multipole moments are obtained from the experimentally observed distributions in each (m,t) bin by means of optimization of maximum likelihood function which takes into account the acceptance of the apparatus [11,39]. In these fits it is usually assumed that moments with $M > 2$ vanish. However, it was pointed out by Sakrejda [16] that moments

up to $M = 4$ may have to be taken into account at larger momentum transfers extending to 1.0 $(\mathrm{GeV/c})^2$.

The expansion coefficients t, p, r, q are simply connected to moments of angular distributions [10]:

$$t^L_M = \epsilon_M < \mathrm{Re}^L_M >= \frac{\epsilon_M}{2\pi} \int I(\Omega, \psi, \delta) \mathrm{Re} Y^L_M(\Omega) d\Omega' \tag{2.3}$$

$$p^L_M = 2\epsilon_M < \cos\psi \mathrm{Re}^L_M >= \frac{2\epsilon_M}{2\pi} \int I(\Omega, \psi, \delta) \mathrm{Re} Y^L_M(\Omega) \cos\psi \cos\delta d\Omega'$$

$$r^L_M = 4 < \sin\psi \mathrm{Im} Y^L_M >= \frac{4}{2\pi} \int I(\Omega, \psi, \delta) \mathrm{Im} Y^L_M \sin\psi \cos\delta d\Omega'$$

$$q^L_M = 4 < \mathrm{Im} Y^L_M >= \frac{4}{2\pi} \int I(\Omega, \psi, \delta) \mathrm{Im} Y^L_M \sin\delta d\Omega'$$

where $d\Omega' = d\Omega d\psi d(-\sin\delta)$. In (2.3), $\epsilon_M = 1$ for $M = 0$ and $\epsilon_M = 2$ for $M \neq 0$. Integrated over the solid angles (θ, ϕ), the distribution (2.1) becomes

$$I(\psi, \delta) = (1 + AP_T \cos\psi) \frac{d^2\sigma}{dmdt} \tag{2.4}$$

where $A = A(s,t,m) = \sqrt{4\pi} p^0_0$ is the polarized target asymmetry analogous to the polarization parameter measured in two-body reactions. In (2.4) $d^2\sigma/dmdt$ is the integrated reaction cross-section

$$\frac{d^2\sigma(s,t,m)}{dmdt} = \int I(\Omega, \psi, \delta) d\Omega' \tag{2.5}$$

Finally we note the relation of moments t^L_M to moments $H(LM)$ introduced by Chung [31,32]:

$$t^L_M = \epsilon_M < \mathrm{Re} Y^L_M >= \epsilon_M \sqrt{\frac{2L+1}{4\pi}} H(LM) \tag{2.6}$$

C. Amplitudes

The reaction $\pi^- p \to \pi^0\pi^0 n$ is described by production amplitudes $H_{\lambda_n,0\lambda_p}(s,t,m,\theta,\phi)$ where λ_p and λ_n are the helicities of the proton and neutron, respectively. The production amplitudes can be expressed in terms of production amplitudes corresponding to definite dimeson spin J using an angular expansion

$$H_{\lambda_n,0\lambda_p} = \sum_{J=0}^{\infty}\sum_{\lambda=-J}^{+J}(2J+1)^{1/2}H^J_{\lambda\lambda_n,0\lambda_p}(s,t,m)d^J_{\lambda 0}(\theta)e^{i\lambda\phi} \tag{2.7}$$

where J is the spin and λ the helicity of the $(\pi^0\pi^0)$ dimeson system. Because of the identity of the two final-state mesons, the "partial waves" with odd J are absent so that $J = 0, 2, 4, \ldots$

In the following we will consider only S-wave ($J = 0$), D-wave ($J = 2$) and G-wave ($J = 4$) amplitudes. Furthermore, we will restrict the dimeson helicity λ to values $\lambda = 0$ or ± 1 only in accordance with the assumption that moments with $M > 2$ vanish. This assumption is supported by experiments.

The "partial wave" amplitudes $H^J_{\lambda\lambda_n,0\lambda_p}$ can be expressed in terms of nucleon helicity amplitudes with definite t-channel exchange naturality. The nucleon s-channel helicity amplitudes describing the production of $(\pi^0\pi^0)$ (or $(\eta\eta)$) system in the S-, D- and G-wave states are:

$$0^-\frac{1}{2}^+ \to 0^+\frac{1}{2}^+ \ : \ H^0_{0+,0+} = S_0, \ H^0_{0+,0-} = S_1 \tag{2.8}$$

$$0^-\frac{1}{2}^+ \to 2^+\frac{1}{2}^+ \ : \ H^2_{0+,0+} = D^0_0, \ H^2_{0+,0-} = D^0_1$$

$$H^2_{\pm 1+,0+} = \frac{D^+_0 \pm D^-_0}{\sqrt{2}}, \ H^2_{\pm 1+,0-} = \frac{D^+_1 \pm D^-_1}{\sqrt{2}}$$

$$0^-\frac{1}{2}^+ \to 4^+\frac{1}{2}^+ \ : \ H^4_{0+,0+} = G^0_0, \ H^4_{0+,0-} = G^0_1$$

$$H^4_{\pm 1+,0+} = \frac{G^+_0 \pm G^-_0}{\sqrt{2}}, \ H^4_{\pm 1+,0-} = \frac{G^+_1 \pm G^-_1}{\sqrt{2}}$$

At large s, the amplitudes S_n, D_n^0, D_n^-, G_n^0, G_n^-, $n = 0,1$ are dominated by unnatural exchanges while the amplitudes D_n^+ and G_n^+, $n = 0,1$ are dominated by natural exchanges. The index $n = |\lambda_p - \lambda_n|$ is nucleon helicity flip.

The observables obtained in experiments on transversely polarized targets in which recoil nucleon polarization is not observed are most simply related to nucleon transversity amplitudes of definite naturality [10,19,40]. For S-, D- and G-waves they are defined as follows:

$$S = k(S_0 + iS_1)\ ,\ \overline{S} = k(S_0 - iS_1) \tag{2.9}$$

$$D^0 = k(D_0^0 + iD_1^0)\ ,\ \overline{D}^0 = k(D_0^0 - iD_1^0)$$

$$D^- = k(D_0^- + iD_1^-)\ ,\ \overline{D}^- = k(D_0^- - iD_1^-)$$

$$D^+ = k(D_0^+ - iD_1^+)\ ,\ \overline{D}^+ = k(D_0^+ + iD_1^+)$$

$$G^0 = k(G_0^0 + iG_1^0)\ ,\ \overline{G}^0 = k(G_0^0 - iG_1^0)$$

$$G^- = k(G_0^- + iG_1^-)\ ,\ \overline{G}^- = k(G_0^- - iG_1^-)$$

$$G^+ = k(G_0^+ - iG_1^+)\ ,\ \overline{G}^+ = k(G_0^+ + iG_1^+)$$

where $k = 1/\sqrt{2}$. The formal proof that the amplitudes defined in (2.9) are actually transversity amplitudes is given from definition in the Appendix in Ref. 19.

The nucleon helicity and nucleon transversity amplitudes differ in the quantization axis for the nucleon spin. The transversity amplitudes S, D^0, D^-, D^+, G^0, G^-, G^+ $(\overline{S}, \overline{D}^0, \overline{D}^-, \overline{D}^+, \overline{G}^0, \overline{G}^-, \overline{G}^+)$ describe the production of the dimeson state with the recoil nucleon spin antiparallel or "down" (parallel or "up") relative to the normal $\vec{n}$ to the production plane. The direction of normal $\vec{n}$ is defined according to Basel convention by $\vec{p}_\pi \times \vec{p}_{\pi\pi}$ where $\vec{p}_\pi$ and $\vec{p}_{\pi\pi}$ are the incident pion and dimeson momenta in the target proton rest frame.

Using the symbols ↑ and ↓ for the nucleon transversities up and down, respectively, the following table shows the spin states of target protons and recoil neutrons and the dimeson helicities corresponding to the transversity amplitudes (2.9):

	p	n	$(\pi^0\pi^0)$
S, D^0, G^0	↑	↓	0
$\overline{S}, \overline{D}^0, \overline{G}^0$	↓	↑	0
D^-, G^-	↑	↓	$+1$ or -1
$\overline{D}^-, \overline{G}^-$	↓	↑	$+1$ or -1
D^+, G^+	↓	↓	$+1$ or -1
$\overline{G}^+, \overline{G}^+$	↑	↑	$+1$ or -1

Parity conservation requires that in the transversity frame the dimeson production with helicities ± 1 depends only on the transversities of the initial and final nucleons. The amplitudes D^-, $\overline{D}^-$, ..., G^+, $\overline{G}^+$ do not distinguish between dimeson helicity states with $\lambda = +1$ or -1. Also, the dimeson production with helicity $\lambda = 0$ is forbidden by parity conservation when the initial and final nucleons have the same transversities.

D. Observables in terms of amplitudes

It is possible to express the moments t^L_M and p^L_M in terms of quantities that do not depend explicitly on whether we use nucleon helicity or nucleon transversity amplitudes. However, eventually we are going to work with transversity amplitudes. The quantities we shall need are spin-averaged partial wave intensity

$$I_A = |A|^2 + |\overline{A}|^2 = |A_0|^2 + |A_1|^2 \tag{2.10}$$

and partial wave polarization

$$P_A = |A|^2 - |\overline{A}|^2 = 2\epsilon_A \mathrm{Im}(A_0 A_1^*) \tag{2.11}$$

where $\epsilon_A = +1$ for $A = S, D^0, D^-, G^0, G^-$ and $\epsilon_A = -1$ for $A = D^+, G^+$. We also introduce spin-averaged interference terms

$$R(AB) = \mathrm{Re}(AB^* + \overline{A}\,\overline{B}^*) = \mathrm{Re}(A_0 B_0 + \epsilon_A \epsilon_B A_1 B_1^*) \tag{2.12}$$

$$Q(AB) = \mathrm{Re}(AB^* - \overline{A}\,\overline{B}^*) = \mathrm{Re}(\epsilon_B A_0 B_1^* - \epsilon_A A_1 B_0^*) \tag{2.13}$$

Then moments t_M^L can be expressed in terms of spin-averaged intensities I_A and spin-averaged interference terms $R(AB)$. The moments p_M^L are then expressed in terms of polarizations P_A and interference terms $Q(AB)$. The formulas for p_M^L are obtained from those for t_M^L using a replacement $I_A \to \epsilon_A P_A$ and $R(AB) \to Q(AB)$ for $\epsilon_A = \epsilon_B = 1$ and $R(AB) \to -Q(AB)$ for $\epsilon_A = \epsilon_B = -1$. There is no mixing of natural and unnatural exchange amplitudes in the moments t_M^L and p_M^L.

Using the results of Lutz and Rybicki [10] and of Chung [32], we obtain the following expressions for moments in terms quantities (2.10)–(2.13) and a constant $c = \sqrt{4\pi}$:

Unpolarized moments (2.14)

$$ct_0^0 = I_S + I_{D^0} + I_{D^-} + I_{D^+} + I_{G^0} + I_{G^-} + I_{G^+}$$

$$ct_0^2 = \sqrt{5}\{\frac{2}{\sqrt{5}}R(SD^0) + \frac{2}{7}I_{D^0} + \frac{1}{7}(I_{D^-} + I_{D^+})$$

$$+\frac{12}{7\sqrt{5}}R(D^0G^0) + \frac{2\sqrt{6}}{7}[R(D^-G^-) + R(D^+G^+)]$$

$$+\frac{20}{77}I_{G^0} + \frac{17}{77}(I_{G^-} + I_{G^+})\}$$

$$ct_1^2 = 2\sqrt{5}\{\frac{2}{\sqrt{10}}R(SD^-) + \frac{\sqrt{2}}{7}R(D^0D^-) + \frac{2\sqrt{3}}{7}R(D^0G^-)$$

$$-\frac{4}{5}\sqrt{\frac{2}{5}}R(D^-G^0) + \frac{2\sqrt{15}}{77}R(G^0G^-)\}$$

$$ct_2^2 = 2\sqrt{5}\{\frac{1}{7}\sqrt{\frac{3}{2}}(I_{D^-} - I_{D^+}) - \frac{1}{7}[R(D^-G^-) - R(D^+G^+)]$$

$$+\frac{5\sqrt{6}}{77}(I_{G^-} - I_{G^+})\}$$

$$ct^4_0 = \sqrt{9}\{\frac{2}{7}I_{D^0} - \frac{4}{21}(I_{D^-} + I_{D^+}) + \frac{2}{3}R(SG^0)+$$

$$+\frac{40\sqrt{5}}{231}R(D^0G^0) + \frac{162}{1001}I_{G^0}+$$

$$+\frac{10}{77}\sqrt{\frac{2}{3}}[R(D^-G^-) + R(D^+G^+)] + \frac{81}{1001}(I_{G^-} + I_{G^+})\}$$

$$ct^4_1 = 2\sqrt{9}\{\frac{2}{7}\sqrt{\frac{5}{3}}R(D^0D^-) + \frac{\sqrt{2}}{3}R(SG^-) + \frac{17\sqrt{10}}{231}R(D^0G^-)$$

$$+\frac{10}{77\sqrt{3}}R(D^-G^0) + \frac{81\sqrt{2}}{1001}R(G^0G^-)\}$$

$$ct^4_2 = 2\sqrt{9}\{\frac{\sqrt{10}}{21}(I_{D^-} - I_{D^+}) + \frac{6\sqrt{15}}{154}[R(D^-G^-) - R(D^+G^+)]$$

$$+\frac{27\sqrt{10}}{1001}(I_{G^-} - I_{G^+})\}$$

$$ct^6_0 = \sqrt{13}\{\frac{30\sqrt{5}}{143}R(D^0G^0) - \frac{20\sqrt{6}}{143}[R(D^-G^-) + R(D^+G^+)]$$

$$+\frac{20}{143}I_{G^0} - \frac{1}{143}(I_{G^-} + I_{G^+})\}$$

$$ct^6_1 = 2\sqrt{13}\{\frac{10\sqrt{21}}{143}R(D^0G^-) + \frac{10\sqrt{35}}{143\sqrt{2}}R(D^-G^0) + \frac{2\sqrt{105}}{143}R(D^0G^-)\}$$

$$ct^6_2 = 2\sqrt{13}\{\frac{4\sqrt{70}}{143}[R(D^-G^-) - R(D^+G^+)] + \frac{\sqrt{105}}{143}(I_{G^-} - I_{G^+})\}$$

$$ct^8_0 = \sqrt{17}\{\frac{490}{2431}I_{G^0} - \frac{392}{2431}(I_{G^-} + I_{G^+})\}$$

$$ct_1^8 = 2\sqrt{17}\{\frac{294\sqrt{5}}{2431}R(G^0G^-)\}$$

$$ct_2^8 = 2\sqrt{17}\{\frac{42\sqrt{35}}{2431}(I_{G^-} - I_{G^+})\}$$

Polarized moments p_M^L (2.15)

$$cp_0^0 = P_S + P_{D^0} + P_{D^-} - P_{D^+} + P_{G^0} + P_{G^-} - P_{G^+}$$

$$cp_0^2 = \sqrt{5}\{\frac{2}{\sqrt{5}}Q(SD^0) + \frac{2}{7}P_{D^0} + \frac{1}{7}(P_{D^-} - P_{D^+}) +$$

$$+\frac{12}{7\sqrt{5}}Q(D^0G^0) + \frac{2\sqrt{6}}{7}[Q(D^-G^-) - Q(D^+G^+)] +$$

$$+\frac{20}{77}P_{G^0} + \frac{17}{77}(P_{G^-} - P_{G^+})\}$$

$$cp_1^2 = 2\sqrt{5}\{\frac{2}{\sqrt{10}}Q(SD^-) + \frac{\sqrt{2}}{7}Q(D^0D^-) + \frac{2\sqrt{3}}{7}Q(D^0G^-)$$

$$-\frac{4}{5}\sqrt{\frac{2}{5}}Q(D^-G^0) + \frac{2\sqrt{15}}{77}Q(G^0G^-)\}$$

$$cp_2^2 = 2\sqrt{5}\{\frac{1}{7}\sqrt{\frac{3}{2}}(P_{D^-} + P_{D^+}) - \frac{1}{7}[Q(D^-G^-) + Q(D^+G^+)]$$

$$+\frac{5\sqrt{6}}{77}(P_{G^-} + P_{G^+})\}$$

$$cp_0^4 = \sqrt{9}\{\frac{2}{7}P_{D^0} - \frac{4}{21}(P_{D^-} - P_{D^+}) + \frac{2}{3}Q(SG^0)$$

$$+\frac{40\sqrt{5}}{231}Q(D^0G^0) + \frac{162}{1001}P_{G^0} +$$

$$+\frac{10}{77}\sqrt{\frac{2}{3}}[Q(D^-G^-) - Q(D^+G^+)] + \frac{81}{1001}(P_{G^-} - P_{G^+})\}$$

$$cp_1^4 = 2\sqrt{9}\{\frac{2}{7}\sqrt{\frac{5}{3}}Q(D^0D^-) + \frac{\sqrt{2}}{3}Q(SG^-) + \frac{17\sqrt{10}}{231}Q(D^0G^-)$$

$$+\frac{10}{77\sqrt{3}}Q(D^-G^0) + \frac{81\sqrt{2}}{1001}Q(G^0G^-)\}$$

$$cp_2^4 = 2\sqrt{9}\{\frac{\sqrt{10}}{21}(P_{D^-} + P_{D^+}) + \frac{6\sqrt{15}}{154}[Q(D^-G^-) + Q(D^+G^+)]$$

$$+\frac{27\sqrt{10}}{1001}(P_{G^-} + P_{G^+})\}$$

$$cp_0^6 = \sqrt{13}\{\frac{30\sqrt{5}}{143}Q(D^0G^0) - \frac{20\sqrt{6}}{143}[Q(D^-G^-) - Q(D^+G^+)]$$

$$+\frac{20}{143}P_{G^0} - \frac{1}{143}(P_{G^-} - P_{G^+})\}$$

$$cp_1^6 = 2\sqrt{13}\{\frac{10\sqrt{21}}{143}Q(D^0G^-) + \frac{10\sqrt{35}}{143\sqrt{2}}Q(D^-G^0) + \frac{2\sqrt{105}}{143}Q(G^0G^-)\}$$

$$cp_2^6 = 2\sqrt{13}\{\frac{4\sqrt{70}}{143}[Q(D^-G^-) + Q(D^+G^+)] + \frac{\sqrt{105}}{143}(P_{G^-} + P_{G^+})\}$$

$$cp_0^8 = \sqrt{17}\{\frac{490}{2431}P_{G^0} - \frac{392}{2431}(P_{G^-} - P_{G^+})\}$$

$$cp_1^8 = 2\sqrt{17}\{\frac{294\sqrt{5}}{2431}Q(G^0G^-)\}$$

$$cp_2^8 = 2\sqrt{17}\{\frac{42\sqrt{35}}{2431}(P_{G^-} + P_{G^+})\}$$

Polarized moments r_M^L (2.16)

$$r_1^2 = 2\sqrt{2}\mathrm{Re}(SD^{+*} - \overline{S}\;\overline{D}^{+*}) + \frac{2\sqrt{10}}{7}\mathrm{Re}(D^0D^{+*} - \overline{D}^0\overline{D}^{+*})$$

$$r_2^2 = \frac{2\sqrt{30}}{7}\mathrm{Re}(D^-D^{+*} - \overline{D}^-\overline{D}^{+*})$$

$$r_1^4 = -\frac{4\sqrt{15}}{7}\mathrm{Re}(D^0 D^{+*} - \overline{D}^0 \overline{D}^{+*})$$

$$r_2^4 = -\frac{4\sqrt{10}}{7}\mathrm{Re}(D^- D^{+*} - \overline{D}^- \overline{D}^{+*})$$

We do not include G-wave contributions in the polarized moments r_M^L. In general, these moments are not well determined in measurements on transversely polarized targets and, as can be seen in Appendix A, the calculation of relative phases between the natural exchange amplitude D^+ and the unnatural exchange amplitudes S, D^0, D^- already involves high degree of ambiguity. The inclusion of G waves would make the situation even less tractable.

III. MODEL INDEPENDENT ANALYSIS OF MEASUREMENTS ON UNPOLARIZED TARGETS.

We will now show that in the mass region where only S- and D-waves dominate, *i.e.,* up to about 1500 MeV, it is possible to perform an analysis of measurements of $\pi^- p \to \pi^0 \pi^0 n$ and $\pi^- p \to \eta\eta n$ on unpolarized targets without the simplifying assumption that production amplitudes do not depend on nucleon spin. However we will find that data on unpolarized targets measure in a model independent way only the partial wave intensities and three unrelated interference phases, and not the production amplitudes which remain undetermined.

When only S- and D-wave contribute, the unpolarized moments are (with $c = \sqrt{4\pi}$):

$$ct_0^0 = I_S + I_{D^0} + I_{D^-} + I_{D^+} \tag{3.1}$$

$$ct_0^2 = 2R(SD^0) + \frac{2\sqrt{5}}{7} I_{D^0} + \frac{\sqrt{5}}{7}(I_{D^-} + I_{D^+})$$

$$ct_1^2 = 2\sqrt{2}R(SD^-) + \frac{2\sqrt{10}}{7}R(D^0 D^-)$$

$$ct_2^2 = \frac{\sqrt{30}}{7}(I_{D^-} - I_{D^+})$$

$$ct_0^4 = \frac{6}{7} I_{D^0} - \frac{4}{7}(I_{D^-} + I_{D^+})$$

$$ct_1^4 = \frac{4\sqrt{15}}{7} R(D^0 D^-)$$

$$ct_2^4 = \frac{2\sqrt{10}}{7}(I_{D^-} - I_{D^+})$$

There are 6 independent observables to determine 7 unknowns – 4 partial wave intensities and 3 spin averaged interference terms. Since there are more unknowns than observables, it is necessary to express the maximum likelihood function $\mathcal{L}$ in terms of the partial wave intensities and the interference terms and fit $\mathcal{L}$ to observed data to find a solution.

For this purpose we will now show that the interference terms $R(AB)$ in (3.1) have a general form

$$R(AB) = \sqrt{I}_A \sqrt{I}_B \cos(\delta_{AB}) \tag{3.2}$$

¿From the definition (2.12) we have

$$R(AB) = \sum_{n=0}^{1} \mathrm{Re}(A_n B_n^*) = \sum_{n=0}^{1} |A_n||B_n| \cos(\phi_n^A - \phi_n^B)$$

We can write

$$R(AB) = \sqrt{I}_A \sqrt{I}_B Z_{AB} \tag{3.3}$$

With definitions for $n = 0, 1$

$$\xi_n^{AB} = \frac{|A_n|}{\sqrt{I}_A} \frac{|B_n|}{\sqrt{I}_B} \ , \ \varphi_n^{AB} = \phi_n^A - \phi_n^B \tag{3.4}$$

we have

$$Z_{AB} = \xi_0^{AB} \cos \varphi_0^{AB} + \xi_1^{AB} \cos \varphi_1^{AB} \tag{3.5}$$

We now recall a theorem from wave theory [41]

$$A_1 \sin(\omega t + \varphi_1) + A_2 \sin(\omega t + \varphi_2) = A \sin(\omega t + \varphi) \tag{3.6}$$

where

$$A^2 = A_1^2 + A_2^2 + 2A_1A_2\cos(\varphi_2 - \varphi_1) \tag{3.7}$$

$$\tan\varphi = \frac{A_1\sin\varphi_1 + A_2\sin\varphi_2}{A_1\cos\varphi_1 + A_2\cos\varphi_2}$$

For $\omega t = \frac{\pi}{2}$ we get

$$A_1\cos\varphi_1 + A_2\cos\varphi_2 = A\cos\varphi \tag{3.8}$$

with A and φ given above. We can apply (3.8) to (3.5) and get

$$Z_{AB} = \xi_{AB}\cos\delta_{AB}$$

where ξ_{AB} and δ_{AB} are given by (3.7) with the appropriate substitutions from (3.5). After some algebra it is possible to show that

$$0 \leq \xi_{AB} \leq +1 \tag{3.9}$$

so that $-1 \leq Z_{AB} \leq +1$. Thus we can actually write $Z_{AB} \equiv \cos\delta_{AB}$ which proves the statement (3.2). The phase δ_{AB} is not simply related to the two relative phases $\phi_0^A - \phi_0^B$ and $\phi_1^A - \phi_1^B$ of the helicity amplitudes $A_n, B_n, n = 0, 1$. Moreover, $\cos\delta_{AB}$ is a measurable parameter along with the intensities I_A and I_B.

We will refer to δ_{SD^0}, δ_{SD^-} and $\delta_{D^0D^-}$ in (3.1) as interference phases. Notice again that interference phases are not relative phases between amplitudes and are thus independent. Whereas relative phases satisfy for $n = 0, 1$

$$(\phi_n^S - \phi_n^{D^0}) + (\phi_n^{D^-} - \phi_n^S) + (\phi_n^{D^0} - \phi_n^{D^-}) = 0 \tag{3.10}$$

there is no such relation for the interference phases.

We can use (3.2) to express the maximum likelihood function $\mathcal{L}$ in terms of the 4 intensities $I_S, I_{D^0}, I_{D^-}, I_{D^+}$ and 3 interference phases δ_{SD^0}, δ_{SD^-} and $\delta_{D^0D^-}$ and fit $\mathcal{L}$ to the observed angular distributions to find a solution for these quantities in each (m, t) bin. We

can conclude that analysis of data on $\pi^- p \to \pi^0\pi^0 n$ unpolarized target is possible without the assumption that production amplitudes are independent of nucleon spin. However the data on unpolarized target cannot determine the 8 moduli and 6 cosines of dependent relative phases of production amplitudes. As we show below, for that determination measurements on polarized target are necessary. The measurements on unpolarized target determine only 4 partial wave intensities and 3 interference phases in a model independent way.

In a mass region where G-waves contribute, measurements on unpolarized target measure 12 independent unpolarized moments t^L_M. There are 7 intensities and 11 spin averaged interference terms in (2.14) for a total of 18 unknowns. In this case model independent amplitude analysis is not possible. However we shall see below that model independent analysis including G-waves is possible for measurements on polarized targets.

IV. COMPARISON WITH MODEL DEPENDENT ANALYSES OF $\pi^- p \to \pi^0\pi^0 n$ ON UNPOLARIZED TARGET.

Both GAMS Collaboration and BNL E852 Collaboration use the assumption of independence of production amplitudes on nucleon spin [31,32] but employ different strategies in actual fits to the observed angular distributions [33,34]. We will confine our discussion to the mass region where S- and D-waves dominate.

The assumption of independence of production amplitudes on nucleon spin means that formally there is one S-wave amplitude S and three D-wave amplitudes D^0, D^-, D^+. The amplitudes have no nucleon spin index. However, as we have argued above, these amplitudes are essentially the single flip helicity amplitudes ($n = 1$) while all helicity non-flip amplitudes ($n = 0$) are assumed to vanish.

In the GAMS approach [33] the unpolarized moments are then written as (with $c = \sqrt{4\pi}$)

$$ct^0_0 = |S|^2 + |D^0|^2 + |D^-|^2 + |D^+|^2 \tag{4.1}$$

$$ct^2_0 = 2\mathrm{Re}(SD^{0*}) + \frac{2\sqrt{5}}{7}|D^0|^2 + \frac{\sqrt{5}}{7}(|D^-|^2 + |D^+|^2)$$

$$ct_1^2 = 2\sqrt{2}\mathrm{Re}(SD^{-*}) + \frac{2\sqrt{10}}{7}\mathrm{Re}(D^0D^{-*})$$

$$ct_2^2 = \frac{\sqrt{30}}{7}(|D^-|^2 - |D^+|^2)$$

$$ct_0^4 = \frac{6}{7}|D^0|^2 - \frac{4}{7}(|D^-|^2 + |D^+|^2)$$

$$ct_1^4 = \frac{4}{7}\sqrt{15}\mathrm{Re}(D^0D^{-*})$$

$$ct_2^4 = \frac{2}{7}\sqrt{10}(|D^-|^2 - |D^+|^2)$$

There are 6 independent equations for 7 unknowns – 4 moduli and 3 cosines of relative phases. The GAMS Collaboration determines these quantities by expressing the maximum likelihood function $\mathcal{L}$ in terms of the amplitudes (moduli and cosines) and fitting $\mathcal{L}$ to the observed angular distribution to find solutions for the moduli and relative phases [27,28,33]. Formally this approach is equivalent to our approach (described in the previous Section) with an additional assumption that the interference phases are not independent but satisfy a constraint

$$\delta_{SD^0} + \delta_{D^-S} + \delta_{D^0D^-} = 0 \tag{4.2}$$

What GAMS Collaboration is actually doing is determining partial wave intensities $I_A, A = S_1, D^0, D^-, D^+$ and interference phases subject to the constraint (4.2). When the constraint (4.2) is removed, their approach becomes fully model independent determination but not of amplitudes but of partial wave intensities.

The BNL E852 employs a different approach [34]. They express the moduli squared and interference terms in (4.1) in terms of real and imaginary parts for amplitudes S, D^0 and D^-. Since there is no interference with D^+, only $|D^+|^2$ is retained. Thus there are 7 unknown quantities. The maximum likelihood function is then expressed in terms of these unknown real and imaginary parts of S, D^0, D^- and $|D^+|^2$ and fitted to the observed angular distributions to find the solution for the amplitudes [34]. Formally this approach is different from our model independent method and relies more explicitly on the assumption that the non-flip helicity amplitudes all vanish.

V. MODEL-INDEPENDENT AMPLITUDE ANALYSIS OF $\pi^- p_\uparrow \to \pi^0\pi^0 n$ MEASURED ON POLARIZED TARGET WITH G-WAVE ABSENT.

In the following we will assume that unpolarized and polarized moments t^L_M and p^L_M (and r^L_M) have been determined using maximum likelihood method in data analysis of measurements of $\pi^- p \to \pi^0\pi^0 n$ and $\pi p \to \eta\eta n$ on polarized targets in a manner previously used in reactions $\pi N_\uparrow \to \pi^-\pi^+ N$ [11–19]. In this Section we show that analytical solution exists for S and D wave in mass region where these waves dominate. In the next section we extend the solution to include the G-wave amplitudes. In both cases we will find it useful to work with nucleon transversity amplitudes (2.9).

In the mass region where S- and D-waves dominate and the G-wave is absent, there are 7 unpolarized moments t^L_M, 7 polarized moments p^L_M and 4 polarized moments r^L_M measured in each (m,t) bin. Looking at equations (2.14) and (2.15), and recalling definitions (2.10)–(2.13), we see that it is advantageous to introduce new observables which are the sum and the difference of corresponding moments t^L_M and p^L_M. We thus define (with $c=\sqrt{4\pi}$) the first set of equations

$$a_1 = \frac{c}{2}(t^0_0 + p^0_0) = |S|^2 + |D^0|^2 + |D^-|^2 + |\overline{D}^+|^2 \tag{5.1}$$

$$a_2 = \frac{c}{2}(t^2_0 + p^2_0) = 2\mathrm{Re}(SD^{0*}) + \frac{2\sqrt{5}}{7}|D^0|^2 + \frac{\sqrt{5}}{7}(|D^-|^2 + |D^+|)$$

$$a_3 = \frac{c}{2}(t^2_1 + p^2_1) = 2\sqrt{2}\mathrm{Re}(SD^{-*}) + \frac{2\sqrt{10}}{7}\mathrm{Re}(D^0 D^{-*})$$

$$a_4 = \frac{c}{2}(t^2_2 + p^2_2) = \frac{\sqrt{30}}{7}(|D^-|^2 - |\overline{D}^+|^2)$$

$$a_5 = \frac{c}{2}(t^4_0 + p^4_0) = \frac{6}{7}|D^0|^2 - \frac{4}{7}(|D^-|^2 + |D^+|^2)$$

$$a_6 = \frac{c}{2}(t^4_1 + p^4_1) = \frac{4}{7}\sqrt{15}\mathrm{Re}(D^0 D^{-*})$$

$$a_7 = \frac{c}{2}(t_2^4 + p_2^4) = \frac{2}{7}\sqrt{10}(|D^-|^2 - |\overline{D}^+|^2)$$

The second set of equations is obtained by defining observables $\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_7$ which are the difference of corresponding moments. We obtain

$$\overline{a}_1 = \frac{c}{2}(t_0^0 - p_0^0) = |\overline{S}|^2 + |\overline{D}^0|^2 + |\overline{D}^-|^2 + |D^+|^2 \tag{5.2}$$

$$\overline{a}_2 = \frac{c}{2}(t_0^2 - p_0^2) = 2\mathrm{Re}(\overline{S}\ \overline{D}^{0*}) + \frac{2\sqrt{5}}{7}|\overline{D}^0|^2 + \frac{\sqrt{5}}{7}(|\overline{D}^-|^2 + |D^+|^2)$$

$$\overline{a}_3 = \frac{c}{2}(t_1^2 - p_1^2) = 2\sqrt{2}\mathrm{Re}(\overline{S}\ \overline{D}^{-*}) + \frac{2\sqrt{10}}{7}\mathrm{Re}(\overline{D}^0\overline{D}^{-*})$$

$$\overline{a}_4 = \frac{c}{2}(t_2^2 - p_2^2) = \frac{\sqrt{30}}{7}(|\overline{D}^-|^2 - |D^+|^2)$$

$$\overline{a}_5 = \frac{c}{2}(t_0^4 - p_0^4) = \frac{6}{7}|\overline{D}^0|^2 - \frac{4}{7}(|\overline{D}^-|^2 + |D^+|^2)$$

$$\overline{a}_6 = \frac{c}{2}(t_1^4 - p_1^4) = \frac{4\sqrt{15}}{7}\mathrm{Re}(\overline{D}^0\overline{D}^{-*})$$

$$\overline{a}_7 = \frac{c}{2}(t_2^4 - p_2^4) = \frac{2}{7}\sqrt{10}(|\overline{D}^-|^2 - |D^+|^2)$$

The first set of 6 independent equations involves 4 moduli $|S|$, $|D^0|$, $|D^-|$, $|\overline{D}^+|$ and 3 cosines of relative phases $\cos(\gamma_{SD^0})$, $\cos(\gamma_{SD^-})$, $\cos(\gamma_{D^0D^-})$. The second set of 6 independent equations involves the amplitudes of opposite transversity – 4 moduli $|\overline{S}|$, $|\overline{D}^0|$, $|\overline{D}^-|$, $|D^+|$ and 3 cosines of their relative phases $\cos(\overline{\gamma}_{SD^0})$, $\cos(\overline{\gamma}_{SD^-})$ and $\cos(\overline{\gamma}_{D^0D^-})$. The two sets are entirely independent and the relative phase between transversity amplitudes up and down is unknown in measurements on transversely polarized targets.

To proceed with the analytical solution, we first find from (5.1)

$$|D^0|^2 = \frac{4}{10}(a_1 - |S|^2) + \frac{7}{10}a_5 \tag{5.3}$$

$$|D^-|^2 = \frac{3}{10}(a_1 - |S|^2) - \frac{7}{10}a_5 + \frac{7}{2\sqrt{30}}a_4$$

$$|\overline{D}^+|^2 = \frac{3}{10}(a_1 - |S|^2) - \frac{7}{10}a_5 - \frac{7}{2\sqrt{30}}a_4$$

$$\cos\gamma_{SD^0} = \frac{1}{|S||D^0|}(A + \frac{1}{2\sqrt{5}}|S|^2) \tag{5.4}$$

$$\cos\gamma_{SD^-} = \frac{1}{|S||D^-|}B$$

$$\cos\gamma_{D^0D^-} = \frac{1}{|D^0||D^-|}C$$

where

$$A = \frac{1}{2}\{a_2 - \frac{1}{\sqrt{5}}a_1 + \frac{1}{2\sqrt{5}}a_5\} \tag{5.5}$$

$$B = \frac{1}{2}\{\frac{1}{\sqrt{2}}a_3 - \frac{1}{2\sqrt{3}}a_6\}$$

$$C = \frac{1}{2}\{\frac{7}{4\sqrt{15}}a_6\}$$

Notice that a_7 is not independent and does not enter in the above equations. Similar solutions can be derived from the second set (5.2) for amplitudes of opposite transversity. However we need one more equation in each set: one equation for $|S|^2$ in the first set and another one for $|\overline{S}|^2$ in the second set.

The additional equations are provided by the relative phases which are not independent:

$$\gamma_{SD^0} - \gamma_{SD^-} + \gamma_{D^0D^-} = (\phi_S - \phi_{D^0}) - (\phi_S - \phi_{D^-}) + (\phi_{D^0} - \phi_{D^-}) = 0 \tag{5.6}$$

$$\overline{\gamma}_{SD^0} - \overline{\gamma}_{SD^-} + \overline{\gamma}_{D^0D^-} = (\overline{\phi}_S - \overline{\phi}_{D^0}) - (\overline{\phi}_S - \overline{\phi}_{D^-}) + (\overline{\phi}_{D^0} - \overline{\phi}_{D^-}) = 0$$

These conditions lead to nonlinear relations between the cosines:

$$\cos^2(\gamma_{SD^0}) + \cos^2(\gamma_{SD^-}) + \cos^2(\gamma_{D^0D^-}) \tag{5.7}$$

$$-2\cos(\gamma_{SD^0})\cos(\gamma_{SD^-})\cos(\gamma_{D^0D^-}) = 1$$

$$\cos^2(\overline{\gamma}_{SD^0}) + \cos^2(\overline{\gamma}_{SD^-}) + \cos^2(\overline{\gamma}_{D^0D^-})$$

$$-2\cos(\overline{\gamma}_{SD^0})\cos(\overline{\gamma}_{SD^-})\cos(\overline{\gamma}_{D^0D^-}) = 1$$

Similar relations also hold for the sines. Next we define combinations of observables

$$D = \frac{4}{10}a_1 - \frac{7}{10}a_5 \tag{5.8}$$

$$E = \frac{3}{10}a_1 - \frac{7}{20}a_5 + \frac{7}{2\sqrt{3}}a_4$$

so that

$$|D^0|^2 = D - \frac{4}{10}|S|^2 \tag{5.9}$$

$$|D^-|^2 = E - \frac{3}{10}|S|^2$$

Substituting into (5.7) first from (5.4) for the cosines and then from (5.9) for $|D^0|^2$ and $|D^-|^2$, we obtain a cubic equation for $x \equiv |S|^2$

$$ax^3 + bx^2 + cx + d = 0 \tag{5.10}$$

where

$$a = \frac{27}{200} \tag{5.11}$$

$$b = \frac{1}{10}(\frac{1}{\sqrt{5}}A - 3D - \frac{9}{2}E)$$

$$c = \frac{1}{10}(3A^2 + 4B^2 - 10C^2 + 2\sqrt{5}BC - 2\sqrt{5}AE + 10DE)$$

$$d = 2ABC - A^2E - B^2D$$

Similar cubic equation can be derived for the amplitude $|\overline{S}|^2$.

Analytical expressions for the 3 roots of the cubic equation (5.10) are given in the Table I of Ref. 21. It is seen from the Table that 3 real solutions exist, one of them is negative and it is rejected. There are in general two positive solutions for $|S|^2$ which lead to two solutions in the Set 1. Similarly there are two solutions in the Set 2 of opposite transversity. Since the two sets are independent there are 4 solutions for partial wave intensities

$$I_A(i,j) = |A(i)|^2 + |\overline{A}(j)|^2 \ , \ \ i,j = 1,2 \tag{5.12}$$

The error propagation in the cubic equation and the calculation of errors on the moduli, cosines and partial wave intensities as well as the treatment of unphysical complex solutions is best handled using the Monte Carlo method described in detail in Ref. 24.

The determination of relative phases between natural exchange amplitude D^+ and unnatural exchange amplitudes S, D^0, D^- is described in the Appendix A.

VI. MODEL-INDEPENDENT AMPLITUDE ANALYSIS OF $\pi^- p_\uparrow \to \pi^0\pi^0 n$ MEASURED ON POLARIZED TARGET WITH G-WAVE INCLUDED.

In the mass region where S-, D- and G-wave all contribute (expected above 1500 MeV), the measurement of $\pi^- p_\uparrow \to \pi^0\pi^0 n$ on polarized target will yield 13 unpolarized moments t^L_M, 13 polarized moments p^L_M and 8 polarized moments r^L_M. Central to our discussion are again the moments t^L_M and p^L_M given by eqs. (2.14) and (2.15). Using the definitions (2.10)–(2.13) we see again that it is useful to define two new sets of observables, one with the sums $t^L_M + p^L_M$ and another one with the differences $t^L_M - p^L_M$. With $c = \sqrt{4\pi}$ we obtain for the first set (sums):

$$a_1 = \frac{c}{2}(t^0_0 + p^0_0) = |S|^2 + |D^0|^2 + |D^-|^2 + |\overline{D}^+|^2 + |G^0|^2 + |G^-|^2 + |\overline{G}^+|^2 \tag{6.1}$$

$$a_2 = \frac{c}{2}(t^2_0 + p^2_0) = \sqrt{5}\{\frac{2}{\sqrt{5}}\mathrm{Re}(SD^{0*}) + \frac{2}{7}|D^0|^2 + \frac{1}{7}(|D^-|^2 + |\overline{D}^+|^2)$$

$$+\frac{12}{7\sqrt{5}}\mathrm{Re}(D^0G^{0*}) + \frac{2\sqrt{6}}{7}[\mathrm{Re}(D^-G^{-*}) + \mathrm{Re}(\overline{D}^+\overline{G}^{+*})]$$

$$+\frac{20}{77}|G^0|^2+\frac{17}{77}(|G^-|^2+|\overline{G}^+|^2)\}$$

$$a_3=\frac{c}{2}(t_1^2+p_1^2)=2\sqrt{5}\{\frac{2}{\sqrt{10}}\mathrm{Re}(SD^{-*})+\frac{\sqrt{2}}{7}\mathrm{Re}(D^0D^{-*})+$$

$$+\frac{2\sqrt{3}}{7}\mathrm{Re}(D^0G^{-*})-\frac{4\sqrt{2}}{5\sqrt{5}}\mathrm{Re}(D^-G^{0*})+\frac{2\sqrt{15}}{77}\mathrm{Re}(G^0G^{-*})\}$$

$$a_4=\frac{c}{2}(t_2^2+p_2^2)=2\sqrt{5}\{\frac{1}{7}\sqrt{\frac{3}{2}}(|D^-|^2-|\overline{D}^+|^2)-$$

$$-\frac{1}{7}[\mathrm{Re}(D^-G^{-*})-\mathrm{Re}(\overline{D}^+\overline{G}^{+*})]+\frac{5\sqrt{6}}{77}(|G^-|^2-|\overline{G}^+|^2)$$

$$a_5=\frac{c}{2}(t_0^4+p_0^4)=\sqrt{9}\{\frac{2}{7}|D^0|^2-\frac{4}{21}(|D^-|^2+|\overline{D}^+|^2)+$$

$$+\frac{2}{3}\mathrm{Re}(SG^{0*})+\frac{40\sqrt{5}}{231}\mathrm{Re}(D^0G^{0*})+$$

$$+\frac{162}{1001}|G^0|^2+\frac{81}{1001}(|G^-|^2+|\overline{G}^+|^2)+$$

$$+\frac{10\sqrt{2}}{77\sqrt{3}}[\mathrm{Re}(D^-G^{-*})+\mathrm{Re}(\overline{D}^+\overline{G}^{+*})]\}$$

$$a_6=\frac{c}{2}(t_1^4+p_1^4)=2\sqrt{9}\{\frac{2}{7}\sqrt{\frac{5}{3}}\mathrm{Re}(D^0D^{-*})+\frac{\sqrt{2}}{3}\mathrm{Re}(SG^{-*})$$

$$+\frac{17\sqrt{10}}{231}\mathrm{Re}(D^0G^{-*})+\frac{10}{77\sqrt{3}}\mathrm{Re}(D^-G^{0*})+\frac{81\sqrt{2}}{1001}\mathrm{Re}(G^0G^{-*})\}$$

$$a_7=\frac{c}{2}(t_2^4+p_2^4)=2\sqrt{9}\{\frac{\sqrt{10}}{21}(|D^-|^2-|\overline{D}^+|^2)+$$

$$+\frac{6\sqrt{15}}{154}[\mathrm{Re}(D^-G^{-*})-\mathrm{Re}(\overline{D}^+\overline{G}^{+*})+\frac{27\sqrt{10}}{1001}(|G^-|^2-|\overline{G}^+|^2)\}$$

$$a_8 = \frac{c}{2}(t_0^6 + p_0^6) = \frac{\sqrt{13}}{143}\{30\sqrt{5}\mathrm{Re}(D^0 G^{0*}) - 20\sqrt{6}[\mathrm{Re}(D^- G^{-*}) +$$

$$+\mathrm{Re}(\overline{D}^+ \overline{G}^{+*})] + 20|G^0|^2 - (|G^-|^2 + |\overline{G}^+|^2)\}$$

$$a_9 = \frac{c}{2}(t_1^6 + p_1^6) = \frac{2\sqrt{13}}{143}\{10\sqrt{21}\mathrm{Re}(D^0 G^{-*}) + \frac{10\sqrt{35}}{\sqrt{2}}\mathrm{Re}(D^- G^{0*})$$

$$+2\sqrt{105}\mathrm{Re}(G^0 G^{-*})\}$$

$$a_{10} = \frac{c}{2}(t_2^6 + p_2^6) = \frac{2\sqrt{13}}{143}\{4\sqrt{70}[\mathrm{Re}(D^- G^{-*}) - \mathrm{Re}(\overline{D}^+ \overline{G}^{+*})]$$

$$+\sqrt{105}(|G^-|^2 - |\overline{G}^+|^2)\}$$

$$a_{11} = \frac{c}{2}(t_0^8 + p_0^8) = \frac{\sqrt{17}}{2431}\{490|G^0|^2 - 392(|G^-|^2 + |\overline{G}^+|^2)\}$$

$$a_{12} = \frac{c}{2}(t_1^8 + p_1^8) = \frac{2\sqrt{17}}{2431}\{294\sqrt{5}\mathrm{Re}(G^0 G^{-*})\}$$

$$a_{13} = \frac{c}{2}(t_2^8 + p_2^8) = \frac{2\sqrt{17}}{2431}\{42\sqrt{35}(|G^-|^2 - |\overline{G}^+|^2)\}$$

The second set of observables $\overline{a}_i, i = 1, 2, \ldots, 13$ is formed similarly by the differences $t_M^L - p_M^L$. It has the same form as set 1 but involves the amplitudes of opposite transversity.

The first set $a_i, i = 1, \ldots, 13$ involves 7 moduli

$$|S|, |D^0|, |D^-|, |\overline{D}^+|, |G^0|, |G^-|, |\overline{G}^+|, \tag{6.2}$$

10 cosines of relative phases between unnatural amplitudes

$$\cos(\gamma_{SD^0}), \cos(\gamma_{SD^-}), \cos(\gamma_{SG^0}), \cos(\gamma_{SG^-}) \tag{6.3}$$

$$\cos(\gamma_{D^0 D^-}), \cos(\gamma_{D^0 G^0}), \cos(\gamma_{D^0 G^-}) \tag{6.4}$$

$$\cos(\gamma_{D^-G^0}), \cos(\gamma_{D^-G^-}), \cos(\gamma_{G^0G^-}) \tag{6.5}$$

and one cosine of relative phase between the two natural amplitudes

$$\cos(\overline{\gamma}_{D^+G^+}) \tag{6.6}$$

The second set $\overline{a}_i, i = 1, \ldots, 13$ involves the same amplitudes but of opposite transversity. We will now show that the cosines (6.4) and (6.5) can be expressed in terms of cosines (6.3). For instance, we can write

$$\gamma_{D^0D^-} = \phi_{D^0} - \phi_{D^-} = (\phi_S - \phi_{D^-}) - (\phi_S - \phi_{D^0}) = \gamma_{SD^-} - \gamma_{SD^0} \tag{6.7}$$

Hence

$$\cos\gamma_{D^0D^-} = \cos\gamma_{SD^0}\cos\gamma_{SD^-} + \sin\gamma_{SD^0}\sin\gamma_{SD^-}$$

Since the signs of the sines $\sin\gamma_{SD^0}$ and $\sin\gamma_{SD^-}$ are not known, we write

$$\sin\gamma_{SD^0} = \epsilon_{SD^0}|\sin\gamma_{SD^0}|\ ,\ \sin\gamma_{SD^-} = \epsilon_{SD^-}|\sin\gamma_{SD^-}| \tag{6.8}$$

Then

$$\cos\gamma_{D^0D^-} = \cos\gamma_{SD^0}\cos\gamma_{SD^-} + \epsilon_{D^0D^-}\sqrt{(1-\cos^2\gamma_{SD^0})(1-\cos^2\gamma_{SD^-})} \tag{6.9}$$

where $\epsilon_{D^0D^-} = \pm 1$ is the sign ambiguity. The remaining cosines in (6.4) and (6.5) can be written in the form similar to (6.9) with their own sign ambiguities. The sign ambiguities of all cosines (6.4) and (6.5) can be written in terms of sign ambiguities $\epsilon_{SD^0}, \epsilon_{SD^-}, \epsilon_{SG^0}, \epsilon_{SG^-}$ corresponding to the sines $\sin\gamma_{SD^0}, \sin\gamma_{SD^-}, \sin\gamma_{SG^0}, \sin\gamma_{SG^-}$. We can write

$$\epsilon_{D^0D^-} = \epsilon_{SD^0}\epsilon_{SD^-} \tag{6.10}$$

$$\epsilon_{D^0G^0} = \epsilon_{SD^0}\epsilon_{SG^0}$$

$$\epsilon_{D^0G^-} = \epsilon_{SD^0}\epsilon_{SG^-}$$

$$\epsilon_{D^-G^0} = \epsilon_{SD^-}\epsilon_{SG^0} \tag{6.11}$$

$$\epsilon_{D^-G^-} = \epsilon_{SD^-}\epsilon_{SG^-}$$

$$\epsilon_{G^0G^-} = \epsilon_{SG^0}\epsilon_{SG^-}$$

First we notice that reversal of all signs $\epsilon_{SD^0}, \epsilon_{SD^-}, \epsilon_{SG^0}$, and ϵ_{SG^-} yields the same sign ambiguities (6.10) and (6.11). Next we notice that the sign ambiguities (6.11) of cosines (6.5) are uniquely determined by the sign ambiguities (6.10) for cosines (6.4). Only sign ambiguities (6.10) are independent and there is 8 sign combinations (6.10). The following table lists all 8 allowed sets of sign ambiguities of cosines (6.4) and (6.4).

	1	2	3	4	5	5	7	8
$\epsilon_{D^0D^-}$	$+$	$-$	$+$	$+$	$-$	$-$	$+$	$-$
$\epsilon_{D^0G^0}$	$+$	$+$	$-$	$+$	$-$	$+$	$-$	$-$
$\epsilon_{D^0G^-}$	$+$	$+$	$+$	$-$	$+$	$-$	$-$	$-$
$\epsilon_{D^-G^0}$	$+$	$-$	$-$	$+$	$+$	$-$	$-$	$+$
$\epsilon_{D^-G^-}$	$+$	$-$	$+$	$-$	$-$	$+$	$-$	$+$
$\epsilon_{G^0G^-}$	$+$	$+$	$-$	$-$	$-$	$-$	$+$	$+$

Using expressions like (6.9) for cosines (6.4) and (6.5), we have 12 unknowns in each nonlinear set of 13 equations $a_i, i = 1, 2, \ldots, 13$ with one choice of sign ambiguities for cosines (6.4) and (6.5) from the above Table. The nonlinear set can be solved numerically or by χ^2 method. In each (m, t) bin we thus have 8 solutions for moduli (6.2) and cosines (6.3)–(6.5), and 8 solutions for amplitudes of opposite transversity from the set $\overline{a}_i, i = 1, 2, \ldots, 13$. Since each solution is uniquely labeled by the choice of sign ambiguities, there is no problem linking solutions in neighbouring (m, t) bins.

Since the 8 solutions from the first set $a_i, i = 1, 2, \ldots, 13$ are independent from the 8 solutions from the second set $\overline{a}_i, i = 1, 2, \ldots, 13$, there will be 64-fold ambiguity in the partial wave intensities. For $A = S, D^0, D^-, D^+, G^0, G^-, G^+$ we can write

$$I_A(i,j) = |A(i)|^2 + |\overline{A}(j)|^2 \ , \ i,j = 1,2,\ldots,8 \tag{6.12}$$

We now will discuss constraints on the moments that should be taken into account at the time of fitting maximum likelihood function $\mathcal{L}$ to the observed angular distribution in the process of constrained optimization.

The observables a_i and $\overline{a}_i, i = 1,2,\ldots,13$ are not all linearly independent. In fact one finds two relations [32]

$$8\sqrt{14}a_4 - 4\sqrt{42}a_7 + \frac{91}{\sqrt{13}}a_{10} - \frac{119}{2}\sqrt{\frac{3}{17}}a_{13} = 0 \tag{6.13}$$

$$8\sqrt{14}\overline{a}_4 - 4\sqrt{42}\overline{a}_7 + \frac{91}{\sqrt{13}}\overline{a}_{10} - \frac{119}{2}\sqrt{\frac{3}{17}}\overline{a}_{13} = 0$$

By adding and subtracting the last two equations we get the same relationship for corresponding moments t^L_M and p^L_M:

$$8\sqrt{14}t^2_2 - 4\sqrt{42}t^4_2 + \frac{91}{\sqrt{13}}t^6_2 - \frac{119}{2}\sqrt{\frac{3}{17}}t^8_2 = 0 \tag{6.14}$$

$$8\sqrt{14}p^2_2 - 4\sqrt{42}p^4_2 + \frac{91}{\sqrt{13}}p^6_2 - \frac{119}{2}\sqrt{\frac{3}{17}}p^8_2 = 0$$

Additional constraints can be obtained by solving for $|G^-|^2+|\overline{G}^+|^2$ from a_{11} and substituting into a_1. Proceeding in the same way also for $|\overline{G}^-|^2 + |G^+|^2$ from $\overline{a}_{11}$ and substituting into $\overline{a}_1$, we get

$$a_1 + \frac{2431}{392\sqrt{17}}a_{11} > 0 \ , \ \overline{a}_1 + \frac{2431}{392\sqrt{17}}\overline{a}_{11} > 0 \tag{6.15}$$

By adding the two inequalities we get

$$t^0_0 + \frac{2431}{392\sqrt{17}}t^8_0 > 0 \tag{6.16}$$

The constraints (6.13) and (6.14), or (6.15) and (6.16), are self-consistency constraints which follow from the assumption that only S-, D- and G-waves contribute. These constraints should be imposed on the maximum likelihood function during the fit to the observed angular distribution. We then deal with constrained optimization [42–44]. A program MINOS 5.0 has been developed at Stanford University for constrained optimization with equalities and inequalities constraints [45].

VII. SUMMARY.

The dependence of hadronic reactions on nucleon spin is now a well-established experimental fact. The measurements of reactions $\pi^- p \to \pi^- \pi^+ n$ and $\pi^+ n \to \pi^+ \pi^- p$ on polarized targets at CERN found a strong dependence of pion production amplitudes on nucleon spin. The assumption thtat pion production amplitudes are independent of nucleon spin is in direct conflict with these experimental findings. The analyses of $\pi^- p \to \pi^0 \pi^0 n$ data based on this assumption thus are not sufficient and may not be fully reliable.

We have shown in Section III that unpolarized data provide model independent information only on the spin averaged partial wave intensities and cosines of three interference phases. To obtain information about the production amplitudes, measurements of $\pi^- p \to \pi^0 \pi^0 n$ on polarized target are necessary. We have shown in Sections V and VI how to perform model independent amplitude analysis of $\pi^- p \to \pi^0 \pi^0 n$ measured on polarized targets. Model independent analysis is possible in the mass region where only S- and D-wave amplitudes contribute, as well as in the mass region where also G-wave amplitudes contribute. Our only assumption was that amplitudes with dimeson helicity $\lambda \geq 2$ do not significantly contribute to the $\pi^0 \pi^0$ production. This assumption is supported by the available data.

On this basis we propose that high statistics measurements of $\pi^- p \to \pi^0 \pi^0 n$ and $\pi^- p \to \eta \eta n$ be made at BNL Multiparticle Spectrometer and at IHEP in Protvino and that model independent amplitude analysis of these reactions be performed. We note that this amplitude analysis will require the unpolarized moments t^L_M which should be determined from the data on unpolarized targets in the same t-bins.

We suggest that the extensions of GAMS and BNL E852 program to measurements on polarized targets will significantly contribute to new developments of hadron spectroscopy on the level of spin dependent production amplitudes and to our understanding of hadron dynamics.

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APPENDIX: CALCULATION OF PHASES γ_{D^+S} AND $\overline{\gamma}_{D^+S}$

In this appendix we solve Eqs. (2.16) for the helicity frame invariant phases $\gamma_{D^+S} = \phi_{D^+} - \phi_S$ and $\overline{\gamma}_{D^+S} = \overline{\phi}_{D^+} - \overline{\phi}_S$. Other phases in (2.16) are then expressed in terms of these phases and the phases (5.4):

$$\gamma_{D^+D^0} = \phi_{D^+} - \phi_{D^0} = (\phi_{D^+} - \phi_S) + (\phi_S - \phi_{D^0}) = \gamma_{D^+S} - \gamma_{D^0S} \tag{A1}$$

$$\gamma_{D^+D^-} = \phi_{D^+} - \phi_{D^-} = (\phi_{D^+} - \phi_S) + (\phi_S - \phi_{D^-}) = \gamma_{D^+S} - \gamma_{D^-S}$$

with similar relations for $\overline{\gamma}_{D^+D^0}$ and $\overline{\gamma}_{D^+D^-}$. The system of equations (2.16) can then be written as

$$b_1 = \frac{7\sqrt{4\pi}}{2\sqrt{30}} r_2^2 = |D^+||D^-| \cos \gamma_{D^+D^-} - |\overline{D}^+||\overline{D}^-| \cos \overline{\gamma}_{D^+D^-} \tag{A2}$$

$$b_2 = \frac{7\sqrt{4\pi}}{4\sqrt{15}} r_1^4 = |D^+||D^0| \cos \gamma_{D^+D^0} - |\overline{D}^+||\overline{D}^0| \cos \overline{\gamma}_{D^+D^0}$$

$$b_3 = \frac{\sqrt{4\pi}}{2\sqrt{2}} r_1^2 - \sqrt{\frac{5}{7}} b_2 = |D^+||S| \cos \gamma_{D^+S} - |\overline{D}^+||\overline{S}| \cos \overline{\gamma}_{D^+S}$$

¿From b_3 we obtain

$$\cos \overline{\gamma}_{D^+S} = \frac{|D^+||S| \cos \gamma_{D^+S} - b_3}{|\overline{D}^+||\overline{S}|} \tag{A3}$$

Using (A1) we obtain from b_2

$$\sin\overline{\gamma}_{D^+S} = -\cos\overline{\gamma}_{D^+S}(\cos\overline{\gamma}_{D^0S}/\sin\overline{\gamma}_{D^0S})+ \tag{A4}$$

$$+\frac{b_2 - |D^+||D^0|(\cos\gamma_{D^+S}\cos\gamma_{D^0S} + \sin\gamma_{D^+S}\sin\gamma_{D^0S})}{|\overline{D}^+||\overline{D}^0|\sin\overline{\gamma}_{D^0S}}$$

We now define

$$c_1 \equiv |D^0||S|\sin\gamma_{D^0S} = \epsilon_1\sqrt{|D^0|^2|S|^2 - (A + \frac{1}{2\sqrt{5}}|S|^2)^2} \tag{A5}$$

$$c_2 \equiv |D^-||S|\sin\gamma_{D^-S} = \epsilon_2\sqrt{|D^-|^2|S|^2 - B^2}$$

$$c_3 = |D^-||D^0|\sin\gamma_{D^-D^0} = \epsilon_3\sqrt{|D^-|^2|D^0|^2 - C^2}$$

where $\epsilon_k = \pm 1$,$k = 1,2,3$ is the ambiguity sign of the sines. The c_3 and the sign ϵ_3 are not independent of c_1 and c_2:

$$|S|^2 c_3 = (A + \frac{1}{2\sqrt{5}}|S|^2)c_2 - Bc_1 \tag{A6}$$

Similarly we define $\overline{c}_1$, $\overline{c}_2$ and $\overline{c}_3$ for amplitudes of opposite transversity. Substituting for $\cos\overline{\gamma}_{D^+S}$ and $\sin\overline{\gamma}_{D^+S}$ from (A3) and (A4) in the equation for b_1 and using the above definitions for c_k, $\overline{c}_k$, $k = 1,2,3$, we obtain

$$(b_1\overline{c}_1 + b_2\overline{c}_2 + b_3\overline{c}_3)|\overline{S}|^2|S| = \tag{A7}$$

$$\sin\gamma_{D^+S}|D^+||\overline{S}|^2(c_1\overline{c}_2 + \overline{c}_1 c_2)+$$

$$+\cos\gamma_{D^+S}|D^+|\{\overline{c}_1(B|\overline{S}|^2 - \overline{B}|S|^2)+$$

$$+\overline{c}_2[(A + \frac{1}{2\sqrt{5}}|S|^2)|\overline{S}|^2 + (\overline{A} + \frac{1}{2\sqrt{5}}|\overline{S}|^2)|S|^2]\}$$

Define

$$d = \frac{b_1\overline{c}_1 + b_2\overline{c}_2 + b_3\overline{c}_3}{c_1\overline{c}_2 + \overline{c}_1 c_2}(\frac{|S|}{|D^+|}) \tag{A8}$$

$$\tan\alpha = \{\bar{c}_1(B|\overline{S}|^2 - \overline{B}|S|^2) + \bar{c}_2[(A + \frac{1}{2\sqrt{5}}|S|^2)|\overline{S}|^2 +$$

$$+(\overline{A} + \frac{1}{2\sqrt{5}}|\overline{S}|^2)|S|^2]\}/(c_1\bar{c}_2 + \bar{c}_1 c_2)|\overline{S}|^2$$

With this notation (A6) takes the form

$$\sin\gamma_{D^+S} + \cos\gamma_{D^+S}\tan\alpha = d \tag{A9}$$

Its solution is

$$\cos\gamma_{D^+S} = \frac{1}{1+\tan^2\alpha}\{d\tan\alpha \pm \sqrt{1+\tan^2\alpha - d^2}\} \tag{A10}$$

$$\sin\gamma_{D^+S} = \frac{1}{1+\tan^2\alpha}\{d \mp \tan\alpha\sqrt{1+\tan^2\alpha - d^2}\}$$

Using (A10) we obtain $\cos\overline{\gamma}_{D^+S}$ and $\sin\overline{\gamma}_{D^+S}$ from (A3) and (A4).

There are four combinations of solutions for moduli $|A|^2$, $|\overline{A}|^2$, $A = S, D^0, D^-, D^+$ entering the calculation of d and $\tan\alpha$. In addition each such combination is accompanied by the fourfold sign ambiguity from the undetermined signs ϵ_k and $\bar{\epsilon}_k$, $k = 1, 2$. This 16-fold ambiguity increases to 32-fold ambiguity due to sign ambiguity in (A10).

The solvability of (A9) imposes a nonlinear constraint on data and on the solution for moduli squared

$$d^2 - 1 \leq \tan^2\alpha \tag{A11}$$

Additional constraints follow from the requirement that cosines and sines of γ_{D^+S} and $\overline{\gamma}_{D^+S}$ have physical values. In principle, these constraints could reduce the overall ambiguity of solution (A10).

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FIGURES

FIG. 1. Polarized target asymmetry in reactions $\pi^- p \to \pi^- \pi^+ n$ and $\pi^+ n \to \pi^+ \pi^- p$. The assumption that the pion production amplitudes do not depend on nucleon spin predicts that polarized target asymmetry be zero.

FIG. 2. The ratio of amplitudes with recoil nucleon transversity “down” and “up” with dimeson helicity $\lambda = 0$ in $\pi^- p \to \pi^- \pi^+ n$ at 17.2 GeV/c and $-t = 0.005 - 0.2$ (GeV/c)^2. The assumption that the pion production amplitudes do not depend on nucleon spin predicts that all ratios be equal to 1. The deviation from unity shows the strength of dependence of production amplitudes on nucleon spin. Based on Fig. 6 of Ref. 14.

FIG. 3. Mass dependence of unnormalized amplitudes $|\overline{S}|^2\Sigma$ and $|S|^2\Sigma$ measured in $\pi^- p_\uparrow \to \pi^- \pi^+ n$ at 17.2 GeV/c at $-t = 0.005 - 0.20$ (GeV/c)^2 using the Monte Carlo method for amplitude analysis (Ref. 24). Both solutions for the transversity “up” amplitude $|\overline{S}|^2\Sigma$ resonate while the transversity “down” amplitude $|S|^2\Sigma$ is nonresonating in both solutions.

FIG. 4. Four solutions for the S-wave intensity I_S measured in the reaction $\pi^- p_\uparrow \to \pi^- \pi^+ n$ at 17.2 GeV/c and $-t = 0.005 - 0.20$ GeV/c using Monte Carlo method for amplitude analysis (Ref. 24). Although both solutions for amplitude $|\overline{S}|^2\Sigma$ resonate, the intensity $I_S(2,2)$ appears nonresonating.

FIG. 5. Definition of the coordinate systems used to describe the target polarization $\vec{P}$ and the decay of the dimeson $\pi^0 \pi^0$ system.